ON VARIATIONAL PROBLEMS OF OPTIMIZATION OF CONTROL PROCESSES

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The problems of optimization of control processes in recent years attract considerable attention of researchers. The classical apparatus of the calculus of variations [1-5] as well as newer methods are employed for their solution. Certain results, important for the theory of optimum systems, have been obtained with the use of the maximum principle of Pontriagin [6-9], the methods of functional analysis [10], and the method of dynamic programming [11].

Numerous questions of optimization of control processes can be formulated in the form of the Lagrange problem [1, 12-15], the Mayer problem [6,8], and the Mayer-Bolza problem of the calculus of variations. Here, the most general of them, the Mayer-Bolza problem, is discussed, with the modifications introduced by the questions of optimization [2-4, 13, 14] and with the limitations imposed on the controls being taken into account. For this case the necessary conditions of minimum are established.

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1. Formulation of the problem. Consider the functions $x_s(t)$ $(s = 1, \ldots, n)$ and $u_k(t)$ $(k = 1, \ldots, m)$ satisfying, for $t_0 < t < T$, the system of n ordinary differential equations of the first order

$$g_s = \dot{x}_s - f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (s = 1, \ldots n) \tag{1.1}$$

and the r finite relations

$$\psi_k = \psi_k(u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r < m) \tag{1.2}$$

and satisfying also the p conditions at the ends (t_0 and T may be not fixed)

$$\varphi_l = \varphi_l [x_1(t_0), \ldots, x_n(t_0), t_0, x_1(T), \ldots, x_n(T), T] = 0$$
$$(l = 1, \ldots, p \leqslant 2n+1) \tag{1.3}$$

It is required to find those functions x_s, u_k for which the functional

$$J = g[x_1(t_0), \ldots, x_n(t_0), t_0, x_1(T), \ldots, x_n(T), T] + \\ + \int_{t_0}^{T} f_0(x_1, \ldots, x_n, u_1, \ldots, u_m, t)\, dt \tag{1.4}$$

assumes the minimum (or maximum) value.

This formulation leads to the Mayer-Bolza problem [13] of a particular type complicated by the existence of Equations (1.2) and the functions $u_k(t)$ whose derivatives do not appear in the equations of the problem. It includes a wide class of optimization problems of control processes [14]. In the discussion of such problems, the functions $u_k(t)$ are called the control parameters or controls, and $x_s(t)$ are called the coordinates. This terminology will be used in the following.

We shall assume that all the requirements of the calculus of variations imposed on the functions used in this formulation are satisfied. We shall investigate only the normal curves [13] of the $n + m$ dimensional space of the coordinates and controls which correspond to a minimum of the functional J. The case of maximum may be reduced to the case being discussed by changing the sign of J, or by changing the signs of the inequalities given in the following.

Unlike in the cases considered previously [14,15], where only the necessary conditions of extremum were discussed, we shall investigate here the trajectories with the general conditions (1.3) for the ends, the functional J including two terms, and we shall state all necessary conditions of minimum of the functional J. In [14] the method of using the equations of the type (1.2) is described with the limitations in the form

$$U_k^{(1)} \leqslant u_k(t) \leqslant U_k^{(2)} \tag{1.5}$$

which determine the interval of admissible changes of the controls. In the present case, the relations (1.2) are assumed in the form

$$\psi_k' = u_k - \chi_k(u_{k_1}) = 0 \tag{1.6}$$

where the function $\chi_k(u_{k_1})$ is defined as

$$\chi_k(u_{k_1}) = \begin{cases} U_k^{(1)}, & \dfrac{d\chi_k}{du_{k_1}} = 0, \quad u_{k_1} \leqslant U_{k_1}^{(1)} \\ \chi_k, & \dfrac{d\chi_k}{du_{k_1}} \neq 0, \quad U_{k_1}^{(1)} < u_{k_1} < U_{k_1}^{(2)} \\ U_k^{(2)}, & \dfrac{d\chi_k}{du_{k_1}} = 0, \quad u_{k_1} \geqslant U_{k_1}^{(2)} \end{cases} \tag{1.7}$$

with u_{k_1} being an additional control parameter. These equations may be also established in the form [4]

$$(U_k^{(1)} - u_k)(u_k - U_k^{(2)}) - u_{k_1}^2 = \psi_k = 0 \tag{1.8}$$

We note that a similar assumption can be used if more general limitations are considered

$$\Omega^{(1)} \leqslant \omega(u_1, \ldots, u_{m'}, t) \leqslant \Omega^{(2)} \tag{1.9}$$

where ψ_k has the form

$$\psi = \omega(u_1, \ldots, u_{m'}, t) - \chi(u_{m'+1}) = 0 \tag{1.10}$$

and $\chi(u_{m+1})$ can be obtained from (1.7) by substituting $\Omega^{(1)}$ and $\Omega^{(2)}$ for $U_k^{(1)}$, and $U_k^{(2)}$.

In this way one additional control is introduced for each of the limitations of the type (1.5) or (1.6). All of them should be included in the total number of m controls existing in the equations of the problem. The additional controls, however, enter only in Equations (1.2) but do not appear in Equations (1.1). The functions f_s contain all the controls if the relations (1.2) reflect a kinematical or other property of the system being optimized, and the optimum problem without limitations is discussed.

With the use of the relations of the type (1.6) or (1.10), the transition is accomplished from the closed domain of the coordinates and the controls actually existing in Equations (1.1) to the open domain of the coordinates and all the controls, including also the additional controls introduced by Equations (1.2).

The limitations (1.5) and (1.9) will be an essential feature of the problems of optimization of control processes. They complicate considerably the solution of the problem since they necessitate the consideration of the discontinuous functions $u_k(t)$. Thus, the functions corresponding to the minimum value of the functional J will be sought among the continuous functions $x_s(t)$ with piece-wise continuous derivatives

$x_s(t)$, and among piece-wise continuous controls $u_k(t)$. In [15], certain conditions of integral type were formulated, which here will not be considered.

2. The condition of extremum of the functional J. In establishing the condition of minimum of the functional J, the equivalent expression is used

$$I = \theta + \int_{t_0}^{T} L\, dt \tag{2.1}$$

where

$$\theta = g + \sum_{l=1}^{p} \rho_l \varphi_l \tag{2.2}$$

$$L = f_0 + \sum_{s=1}^{n} \lambda_s g_s - \sum_{k=1}^{r} \mu_k \psi_k = \sum_{s=1}^{n} \lambda_s \dot{x}_s - H \tag{2.3}$$

$$H = H_\lambda + H_\mu = \sum_{s=0}^{n} \lambda_s f_s + \sum_{k=1}^{r} \mu_k \psi_k \qquad (\lambda_0 = -1) \tag{2.4}$$

Here, ρ_l, $\lambda_s(t)$, and $\mu_k(t)$ are the undetermined multipliers of Lagrange which are to be calculated. Furthermore, the first variation ΔI of the functional I is constructed and assumed to be equal to zero. This results in the sought condition of extremum of the functional J.

Such a procedure, although for a simpler problem, has been described in [14]. Therefore, only the final results will be given:

The equations

$$\dot{\lambda}_s^{\pm} = -\frac{\partial H}{\partial x_s^{\pm}} \qquad (s = 1, \ldots, n), \qquad \frac{\partial H}{\partial u_k^{\pm}} = 0 \qquad (k = 1, \ldots, m) \tag{2.5}$$

the boundary conditions

$$\lambda_s^-(t_0) - \frac{\partial \theta}{\partial x_s(t_0)} = 0 \qquad (s = 1, \ldots, n), \qquad (f_0)_{t_0} - \frac{d\theta}{dt_0} = 0 \tag{2.6}$$

$$\lambda_s^+(T) + \frac{\partial \theta}{\partial x_s(T)} = 0 \qquad (s = 1, \ldots, n), \qquad (f_0)_{t_0} + \frac{d\theta}{dT} = 0 \tag{2.7}$$

The Erdmann-Weierstrass conditions (i.e. the conditions of continuity of $\lambda_s(t)$ and H)

$$\lambda_s^-(t^*) = \lambda_s^+(t^*) \qquad (s = 1, \ldots, n), \qquad (H^-)_{t^*} = (H^+)_{t^*} \tag{2.8}$$

The derivatives $d\theta/dt_0$ and $d\theta/dt$ in Equations (2.6) and (2.7) are equal to

$$\frac{d\theta}{dt_0} = \frac{\partial\theta}{\partial t_0} + \sum_{s=1}^{n} \frac{\partial\theta}{\partial x_s(t_0)} \dot{x}_s(t_0), \qquad \frac{d\theta}{dT} = \frac{\partial\theta}{\partial T} + \sum_{s=1}^{n} \frac{\partial\theta}{\partial x_s(T)} \dot{x}_s(T)$$

As was done in [14], we assume that, in the interval $t_0 \leqslant t \leqslant T$, there is one point $t = t^*$ of discontinuity of the controls $u_k(t)$, and we denote by the signs − and + the values of the corresponding functions in the sub-intervals $t_0 \leqslant t \leqslant t^*$ and $t^* \leqslant t \leqslant T$.

The relations (2.5) to (2.8) represent the conditions of extremum of the functional J. In order to solve the problem of optimization, they should be complemented by Equations (1.1) and (1.2), which can be written in the form

$$\dot{x}_s^{\pm} = \frac{\partial H}{\partial \lambda_s^{\pm}} \quad (s = 1, \ldots, n), \qquad \frac{\partial H}{\partial \mu_k^{\pm}} = 0 \quad (k = 1, \ldots, r) \tag{2.9}$$

by the end conditions (1.3), and by the conditions of continuity of the coordinates

$$x_s^-(t^*) = x_s^+(t^*) \qquad (s = 1, \ldots, n) \tag{2.10}$$

Now, to determine the $4n + 2m + 2r$ functions $x_s^{\pm}(t)$, $\lambda_s^{\pm}(t)$, $u_k^{\pm}(t)$, and $\mu_k^{\pm}(t)$, we have the $2n + 2m$ Equations (2.5) and the $2n + 2r$ Equations (2.9). Integration of the differential equations introduces $4n$ constants, which can be determined together with the multipliers $\rho_l (l = 1, \ldots, p)$ and the quantities t_0, t^*, and T from the $4n + p + 3$ conditions (2.6) to (2.8), (2.10).

The second group of Equations (2.5) should be noted; they coincide with the necessary conditions of extremum of the function H with respect to the controls u_k. If the functions f_s and ψ_k do not depend explicitly on time, the following first integral exists

$$H = H_\lambda + H_\mu = h = \text{const} \tag{2.11}$$

In this case the relations are valid

$$-\frac{\partial\theta}{\partial t_0} = \frac{\partial\theta}{\partial T} = h \tag{2.12}$$

which replace the second group of Equations (2.6) and (2.7).

3. The necessary condition of Weierstrass. The necessary and sufficient condition of Weierstrass for the absolute minimum of the functional J can be established with the use of the Weierstrass function E, which in this case has the form

$$E = L(x_1, \ldots, x_n, \dot{X}_1, \ldots, \dot{X}_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - L(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - \sum_{s=1}^{n} (\dot{X}_s - \dot{x}_s) \frac{\partial L}{\partial \dot{x}_s} \tag{3.1}$$

Here, x_s and u_k correspond to the curve for which the functional J is minimum; X_s and U_k are arbitrary admissible functions satisfying Equations (1.1), (1.2), and the conditions (1.3).

In the textbooks [12,13] a different form of the function E is given, without the controls u_k and U_k, which is compatible with the problems discussed in those books. The necessity of introducing the controls u_k and U_k into the variational problems of optimization of control processes may be shown by repeating the arguments and calculations leading to the necessary condition of Weierstrass [13] and taking into account that the functions f_s and ψ_k depend on the controls u_k but do not depend on their derivatives (this is given in the Appendix, Section 6).

Substituting expression (2.3) into Equation (3.1), we obtain

$$E = -H(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) + \\ + H(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) \tag{3.2}$$

The necessary condition of a strong minimum of the functional J

$$E \geqslant 0 \tag{3.3}$$

is equivalent to the inequality

$$H(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) \leqslant \\ \leqslant H(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) \tag{3.4}$$

Exceeding the scope of this paper, we can note that the sufficient condition of Weierstrass for the absolute minimum follows from (3.4) if the sign of equality is omitted. Thus, in an optimum system corresponding to a minimum value of the functional J, the function H is maximum with respect to the controls u_k, for their arbitrary admissible values.

Since the additional controls, discussed in Section 1, do not enter into the function H_λ and since $H_\mu \equiv 0$, the Weierstrass condition and the extremum condition for the problems discussed can be formulated in a form analogous to the maximum principle of Pontriagin [6,8].

The parameters of control u_k, for which the functional J reaches its minimum value, correspond also to the maximum of the function H_λ for arbitrary admissible $x_s(t)$, $\lambda_s(t)$, $\mu_k(t)$ satisfying Equations (2.9), (2.5), conditions (1.3), (2.6), (2.7), and the continuity conditions

(2.8), (2.10).

It should be stressed that with the use of this principle, important results of the theory of optimum systems have been obtained in the papers by Pontriagin, Gamkrelidze, Boltianskii, Rozonoer, and others.

4. The necessary condition of Clebsch. In order to derive the necessary condition of Clebsch for a weak minimum of the functional J, we can use the results of the preceding section. We assume that U_k and $\dot{X}_s$, satisfying Equations (1.1) and (1.2), differ from u_k and $\dot{x}_s$ by small quantities, such that

$$U_k = u_k + \delta u_k, \qquad \dot{X}_s = \dot{x}_s + \delta \dot{x}_s \tag{4.1}$$

where δu_k and $\delta \dot{x}_s$ are small admissible variations satisfying the variational equations along the curve corresponding to a minimum value of the functional J

$$\delta \dot{x}_s - \sum_{k=1}^{m} \frac{\partial f_s}{\partial u_k} \delta u_k = 0 \qquad (s = 1, \ldots, n) \tag{4.2}$$

$$\sum_{\beta=1}^{m} \frac{\partial \psi_k}{\partial u_\beta} \delta u_\beta = 0 \qquad (k = 1, \ldots, r) \tag{4.3}$$

We substitute expressions (4.1) into Equation (3.1) and we expand the first term of its right-hand side in a series in δu_k and $\delta \dot{x}_s$. We have then

$$E = \sum_{s=1}^{n} \sum_{\alpha=1}^{n} \frac{\partial^2 L}{\partial \dot{x}_s \partial \dot{x}_\alpha} \delta \dot{x}_s \delta \dot{x}_\alpha + 2 \sum_{s=1}^{n} \sum_{k=1}^{m} \frac{\partial^2 L}{\partial \dot{x}_s \partial u_k} \delta \dot{x}_s \delta u_k + \sum_{k=1}^{m} \sum_{\beta=1}^{m} \frac{\partial^2 L}{\partial u_k \partial u_\beta} \delta u_k \delta u_\beta \tag{4.4}$$

where the terms of the order higher than two are neglected. Substituting L from (2.3) into this last relation, and using the condition (3.3), we obtain

$$\sum_{k=1}^{m} \sum_{\beta=1}^{m} \frac{\partial^2 H}{\partial u_k \partial u_\beta} \delta u_k \delta u_\beta \leqslant 0 \tag{4.5}$$

This inequality, together with equations (4.2) and (4.3), represents the necessary condition of Clebsch for a weak minimum of the functional J. It is easy to see that this condition and the extremum condition coincide with the necessary condition for a maximum of H_λ with respect to the controls u_k, with Equations (1.2) being satisfied and for small admissible variations of the controls. All the derivatives in Equations (4.2) to (4.5) are to be calculated at the points of the curve corresponding to a minimum of the functional J.

5. The necessary condition of Jacobi. The last necessary condition of minimum of the functional J is the condition of Jacobi. According to this condition we require that the second variation $\Delta^2 I$ of the functional I, calculated at the curve corresponding to a minimum of the functional J, does not assume negative values [13]. This variation is of the following form

$$\Delta^2 I = 2\varphi[\Delta x_1(t_0), \ldots, \Delta x_n(t_0), \delta t_0, \Delta x_1(T), \ldots, \Delta x_n(T), \delta T] + \\ + 2\sum_{s=1}^{n}\left[\frac{\partial L}{\partial x_s}\Delta x_s \delta t\right]_{t_0}^{T} + \left\{\left[\frac{\partial L}{\partial t} - \sum_{s=1}^{n}\frac{\partial L}{\partial x_s}\dot{x}_s\right]\delta t^2\right\}_{t_0}^{T} + \\ + \int_{t_0}^{T} 2\omega(\delta x_1, \ldots, \delta x_n, \delta u_1, \ldots, \delta u_m)\, dt \tag{5.1}$$

and it can be determined as described in the book by Bliss [13].

In the Expression (5.1), 2ϕ and 2ω denote the quadratic forms

$$2\varphi = \left\{\sum_{s=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial^2\theta}{\partial x_s \partial x_\alpha}\Delta x_s \Delta x_\alpha + 2\sum_{s=1}^{n}\frac{\partial^2\theta}{\partial t\,\partial x_s}\Delta x_s \delta t + \frac{\partial^2\theta}{\partial t^2}\delta t^2\right\}_{t_0}^{T} \tag{5.2}$$

$$2\omega = \sum_{s=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial^2 L}{\partial x_s \partial x_\alpha}\delta x_s \delta x_\alpha + 2\sum_{s=1}^{n}\sum_{k=1}^{m}\frac{\partial^2 L}{\partial x_s \partial u_k}\delta x_s \delta u_k + \sum_{k=1}^{m}\sum_{\beta=1}^{m}\frac{\partial^2 L}{\partial u_k \partial u_\beta}\delta u_k \delta u_\beta \tag{5.3}$$

The intervals used at certain terms in the relation (5.1) and the form (5.2) are introduced to simplify the notations, and they denote, for example in the case of the second term in equation (5.1)

$$2\sum_{s=1}^{n}\left[\frac{\partial L}{\partial x_s}\Delta x_s \delta t\right]_{t_0}^{T} = 2\sum_{s=1}^{n}\left(\frac{\partial L}{\partial x_s}\right)_T \Delta x_s(T)\,\delta T - 2\sum_{s=1}^{n}\left(\frac{\partial L}{\partial x_s}\right)_{t_0}\Delta x_s(t_0)\,\delta t_0$$

Finally, $\Delta x_s(t_0)$ and $\Delta x_s(T)$ denote the variations of the ends of the comparison curves

$$\Delta x_s(t_0) = \delta x_s(t_0) + \dot{x}_s(t_0)\,\delta t_0, \quad \Delta x_s(T) = \delta x_s(T) + \dot{x}_s(T)\,\delta T \tag{5.4}$$

All the coefficients of the variations in Equations (5.1) to (5.3) are to be calculated at the points of the curve corresponding to a minimum of the functional J.

Using the relation (2.3), we can express the coefficients of the quadratic from (5.3) in terms of the second derivatives of the function H, which results in

$$-2\omega = \sum_{s=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial^2 H}{\partial x_s \partial x_\alpha}\delta x_s\, \delta x_\alpha + 2\sum_{s=1}^{n}\sum_{k=1}^{m}\frac{\partial^2 H}{\partial x_s \partial u_k}\delta x_s\, \delta u + \sum_{k=1}^{m}\sum_{\beta=1}^{m}\frac{\partial^2 H}{\partial u_k \partial u_\beta}\delta u_k\, \delta u_\beta \tag{5.5}$$

Let us consider now the condition of non-negativeness of the second variation (5.1). It can be obtained by solving the associated problem of minimum of the second variation [13], i.e. by determining such variations $\delta x_1, \ldots, \delta x_n, \delta u_1, \ldots, \delta u_m, \delta t_0$, and δT, related by the variational equations at the curve corresponding to a minimum of the functional J

$$\delta g_s = \delta \dot{x}_s - \sum_{\alpha=1}^{n} \frac{\partial f_s}{\partial x_\alpha} \delta x_\alpha - \sum_{k=1}^{m} \frac{\partial f_s}{\partial u_k} \delta u_k = 0 \qquad (s = 1, \ldots, n) \qquad (5.6)$$

$$\delta \psi_k = \sum_{\alpha=1}^{n} \frac{\partial \psi_k}{\partial x_\alpha} \delta x_\alpha + \sum_{\beta=1}^{m} \frac{\partial \psi_k}{\partial u_\beta} \delta u_\beta = 0 \qquad (k = 1, \ldots, r) \qquad (5.7)$$

and the conditions

$$\frac{d\varphi_l}{dt_0} \delta t_0 + \sum_{s=1}^{n} \frac{\partial \varphi_l}{\partial x_s(t_0)} \delta x_s(t_0) + \frac{d\varphi_l}{dT} \delta T + \sum_{s=1}^{n} \frac{\partial \varphi_l}{\partial x_s(T)} \delta x_s(T) = 0 \quad (l = 1, \ldots, p) \qquad (5.8)$$

that the second variation $\Delta^2 I$ reaches its minimum value. In this case, $\Delta^2 I$ can be represented in the following form

$$\Delta^2 I = j\,[\delta x_1(t_0), \ldots, \delta x_n(t_0), \delta t_0, \delta x_1(T), \ldots, \delta x_n(T), \delta T] + \\ + \int_{t_0}^{T} 2\omega(\delta x_1, \ldots, \delta x_n, \delta u_1, \ldots, \delta u_m)\, dt \qquad (5.9)$$

which follows from the substitution of the variations (5.4) into (5.1).

The form of the functional (5.9) and the restrictions (5.6) to (5.8) indicate that the associated problem of minimum of the second variation reduces to the variational problem of Mayer and Bolza, of the type described in Section 1. In the solution of this problem, the results described above may be used.

It should be noted that in many cases we may limit ourselves to the investigation of the extremum conditions and the conditions of Weierstrass or Clebsch. The associated problem of minimum of the second variation $\Delta^2 I$ of the functional I has actually the trivial solution

$$\delta x_1 = \ldots = \delta x_n = \delta u_1 = \ldots = \delta u_m = \delta t_0 = \delta T = 0 \qquad (5.10)$$

If this solution proves to be unique and satisfying the condition of extremum of the second variation, then the condition of Jacobi is thus fulfilled.

6. Appendix. The necessary condition of Weierstrass. In establishing the necessary condition of Weierstrass for an absolute minimum of the functional J, we shall follow the book by Bliss [13]. The variational problem will be considered as formulated in Section 1. We assume that the functions f_s and ψ_k have derivatives of the order used in the following discussion.

In the $n + m$ dimensional space $x_1, \ldots, x_n, u_1, \ldots, u_m$ we shall consider a normal [13] curve C, satisfying Equations (1.1), (1.2), and conditions (1.3), which corresponds to the minimum of the functional J. We shall assume that the matrix

$$\frac{\partial \psi}{\partial u} = \left\| \frac{\partial \psi_i}{\partial u_j} \right\|$$

whose i, jth element is the derivative $\partial \psi_i / \partial u_j$, has on the curve C the rank equal to r, i.e. equal to the number of equations (1.2).

Then, repeating the calculations presented in the book by Bliss, we find that the curve C may be included in the p-parameter family of curves

$$x_s(b, t) \qquad (s = 1, \ldots, n), \qquad u_k(b, t) \qquad (k = 1, \ldots, m) \tag{6.1}$$

satisfying Equations (1.1) and (1.2), with the values $b_1 = \ldots = b_p = 0$ corresponding to the curve C. Here and in the following, to simplify notations, b denotes the whole set of parameters $b_1, \ldots, b_p$. Similar notation x and u will be used for the set of coordinates $x_1, \ldots, x_n$ and controls $u_1, \ldots, u_m$.

In the interval $t_0 \leqslant t \leqslant T$, we select now a point t', not coinciding with a corner point of the curve C, and we construct three families of curves

$$\begin{gathered}
x_s(b, t), \qquad u_k(b, t) \qquad (t_0 - \delta < t < t', \ |b| < \varepsilon) \\
X_s(b, t), \qquad U_k(t) \qquad (t' \leqslant t \leqslant t' + e, \ |b| < \varepsilon, \ |e| < \varepsilon) \\
x_s(b, e, t), \qquad u_k(b, t) \qquad (t' + e \leqslant t < T + \delta, \ |b| < \varepsilon, \ |e| < \varepsilon) \\
(s = 1, \ldots, n, \quad k = 1, \ldots, m)
\end{gathered} \tag{6.2}$$

satisfying the equations

$$\dot{x}_s - f_s(x, u, t) = 0 \qquad (s = 1, \ldots, n), \qquad \psi_k(u, t) = 0 \qquad (k = 1, \ldots, r) \tag{6.3}$$

in the first and the third intervals.

The curves of the second family satisfy the equations

$$\dot{X}_s - f_s(X, U, t) = 0 \qquad (s = 1, \ldots, n), \qquad \psi_k(U, t) = 0 \qquad (k = 1, \ldots, r) \tag{6.4}$$

In constructing these families of curves, the conditions have been used

$$\begin{gathered}
x_s(b, t_0) = x_s(t_0) + \sum_{\alpha=1}^{p} b_\alpha \xi_{s\alpha}(t_0) \\
X_s(b, t') = x_s(b, t'), \qquad x_s(b, e, t' + e) = X_s(b, t' + e) \qquad (s = 1, \ldots, n)
\end{gathered} \tag{6.5}$$

$U_k(t)$ denote arbitrary admissible functions. For $b = e = 0$, the first family and the third family yield the functions determining the curve C. We use the notation

$$\xi_{s\alpha} = \frac{\partial x_s}{\partial b_\alpha}, \qquad \zeta_{k\alpha} = \frac{\partial u_k}{\partial b_\alpha} \tag{6.6}$$

for the variations of the first and the third families with respect to the parameters b_α, and the notation

$$\xi_s = \frac{\partial x_s}{\partial e}, \qquad \zeta_k = \frac{\partial u_k}{\partial e} \equiv 0 \tag{6.7}$$

for similar variations with respect to the parameter e. From (6.2) and (6.5) we have

$$\xi_s(t) = 0 \quad (t_0 - \delta < t < t'), \qquad \dot{x}_s(t') + \xi_s(t') = \dot{X}_s(t') \tag{6.8}$$

We introduce the variations $\tau_{0\alpha}$ and $\tau_{T\alpha}$ of the abscissas of the left-hand and the right-hand end with respect to the parameters b_α

$$t_0(b) = t_0 + \sum_{\alpha=1}^{p} b_\alpha \tau_{0\alpha}, \qquad T(b) = T + \sum_{\alpha=1}^{p} b_\alpha \tau_{T\alpha} \tag{6.9}$$

where t_0 and T correspond to the curve C. Substituting the functions x_s and u_k, and the quantities t_0 and T from Equations (6.2) and (6.9) into the boundary conditions (1.3), we obtain the equations

$$\varphi_l = \varphi_l(b, e) = \varphi_l\{x[b, t_0(b)], t_0(b), x[b, e, T(b)], T(b)\} = 0 \qquad (l = 1, \ldots, p) \tag{6.10}$$

We note that the determinant $|\partial \phi_l / \partial b_\alpha|$ is different from zero on the curve C, as this curve is considered to be normal. But then Equations (6.10) have the solutions

$$b_\alpha = B_\alpha(e) \tag{6.11}$$

which become equal to zero for $e = 0$.

Eliminating with the use of (6.11) the parameters b_α from the expressions (6.2), we obtain a one-parameter family of curves satisfying Equations (1.1) and (1.2) and the end conditions (1.3). This family contains the curve C at $e = 0$. For this value of $e = 0$, the equation holds

$$\sum_{\alpha=1}^{p} \left(\frac{\partial \varphi_l}{\partial b_\alpha}\right)_0 B_\alpha'(0) + \left(\frac{\partial \varphi_l}{\partial e}\right)_0 = 0 \tag{6.12}$$

Here, the subscript 0 indicates that the values of the derivatives are calculated for $e = 0$.

Since the curve C corresponds to a minimum of the functional J, its derivative with respect to e ($e > 0$), for $e = 0$, cannot be negative.

Consequently, the necessary condition of minimum of the functional J will be the inequality

$$\left(\frac{dJ}{de}\right)_0 = \sum_{\alpha=1}^{p} \left(\frac{\partial J}{\partial b_\alpha}\right)_0 B_\alpha'(0) + \left(\frac{\partial J}{\partial e}\right)_0 \geqslant 0 \tag{6.13}$$

Calculating the values of the derivatives in this inequality, we substitute the functions (6.2) into the functional J, and we obtain

$$J(b, e) = J[x(b, t_0(b)), \quad t_0(b), x(b, e, T(b)), T(b)] \tag{6.14}$$

To the right-hand side of this relation we now add a component identically equal to zero

$$\int_{t_0}^{T} (L - f_0)\, dt$$

Thus, we have the sum

$$\begin{aligned} J = g(b, e) &+ \int_{t_0(b)} L[x(b, t), \dot{x}(b, t), u(b, t), \lambda(t), \mu(t), t]\, dt + \\ &+ \int_{t'}^{t'+e} L[X(b, t), \dot{X}(b, t), U(t), \lambda(t), \mu(t), t]\, dt + \\ &+ \int_{t'+e}^{T(b)} L[x(b, e, t), \dot{x}(b, e, t), u(b, t), \lambda(t), \mu(t), t]\, dt \end{aligned} \tag{6.15}$$

Differentiating it with respect to b_α

$$\begin{aligned} \frac{\partial J}{\partial b_\alpha} = &\sum_{s=1}^{n} \left[\frac{\partial g}{\partial x_s(t_0)} - \left(\frac{\partial L}{\partial \dot{x}_s}\right)_{t_0}\right] \xi_{s\alpha}(t_0) + \sum_{s=1}^{n} \left[\frac{\partial g}{\partial x_s(T)} + \left(\frac{\partial L}{\partial \dot{x}_s}\right)_T\right] \xi_{s\alpha}(T) + \\ &+ \frac{dg}{dt_0}\tau_{0\alpha} + \frac{dg}{dT}\tau_{T\alpha} + \sum_{s=1}^{n} \left[\left(\frac{\partial L}{\partial \dot{x}_s}\right)_{t'+e} \xi_{s\alpha}(t'+e) - \left(\frac{\partial L}{\partial \dot{x}_s}\right)_{t'} \xi_{s\alpha}(t')\right] + \\ &+ \int_{t_0(b)}^{t'} \left\{\sum_{s=1}^{n} \left[\frac{\partial L}{\partial x_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_s}\right)\right] \xi_{s\alpha} + \sum_{\beta=1}^{m} \frac{\partial L}{\partial u_\beta} \zeta_{\beta\alpha}\right\} dt + \\ &+ \int_{t'}^{t'+e} \left\{\sum_{s=1}^{n} \frac{\partial L}{\partial X_s} \frac{\partial X_s}{db_\alpha} + \frac{\partial L}{\partial \dot{X}_s} \frac{\partial \dot{X}_s}{\partial b_\alpha}\right\} dt + \\ &+ \int_{t'+e}^{T(b)} \left\{\sum_{s=1}^{n} \left[\frac{\partial L}{\partial x_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_s}\right)\right] \xi_{s\alpha} + \sum_{\beta=1}^{m} \frac{\partial L}{\partial u_\beta} \zeta_{\beta\alpha}\right\} dt \end{aligned} \tag{6.16}$$

Calculating the value of this derivative for $e = 0$, we note that for $e = 0$ the second integral on the right-hand side becomes equal to zero. Assuming that the conditions of extremum are satisfied, we see that only the first four terms of the sum (6.16) are different from zero. They can be transformed to the form

$$\sum_{s=1}^{n}\left\{\left[\frac{\partial g}{\partial x_s(t_0)}-\left(\frac{\partial L}{\partial \dot{x}_s}\right)_{t_0}\right]\xi_{s\alpha}(t_0)+\left[\frac{\partial g}{\partial x_s(T)}+\left(\frac{\partial l}{\partial \dot{x}_s}\right)_T\right]\xi_{s\alpha}(T)\right\}+$$

$$+\frac{dg}{dT}\tau_{T\alpha}+\frac{dg}{dt_0}\tau_{0\alpha}=-\sum_{l=1}^{p}\rho_l\frac{\partial\varphi_l}{\partial b_\alpha} \tag{6.17}$$

and we have finally

$$\left(\frac{\partial J}{\partial b_\alpha}\right)_0+\sum_{l=1}^{p}\rho_l\left(\frac{\partial\varphi_l}{\partial b_\alpha}\right)_0=0 \tag{6.18}$$

In a similar way the derivative is determined

$$\left(\frac{\partial J}{\partial e}\right)_0=\sum_{s=1}^{n}\frac{\partial g}{\partial x_s(T)}\xi_s(T)+L(x,\dot{X},U,\lambda,\mu,t)_{t'}-L(x,\dot{x},u,\lambda,\mu,t)_{t'}-$$

$$-\sum_{s=1}^{n}\left(\frac{\partial L}{\partial \dot{x}_s}\right)_{t'}\xi_s(t')+\int_{t'+e}^{T(b)}\sum_{s=1}^{n}\left[\frac{\partial L}{\partial x_s}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_s}\right)\right]\xi_s\,dt \tag{6.19}$$

The last term of the right-hand side of this expression becomes equal to zero according to the condition of extremum. Considering that the variations of the abscissas of both ends of the family are equal to zero, $\partial t_0/\partial e = \partial T/\partial e = 0$, we obtain the following result

$$\left(\frac{\partial J}{\partial e}\right)_0+\sum_{l=1}^{p}\rho_l\left(\frac{\partial\varphi_l}{\partial e}\right)_0=(E)_{t'} \tag{6.20}$$

where E denotes the Weierstrass function of our problem

$$E=L(x,\dot{X},U,\lambda,\mu,t)-L(x,\dot{x},u,\lambda,\mu,t)-\sum_{s=1}^{n}(\dot{X}_s-\dot{x}_s)\frac{\partial L}{\partial \dot{x}_s} \tag{6.21}$$

Substituting the derivatives $(\partial J/\partial b_\alpha)_0$ and $(\partial J/\partial e)_0$ into the inequality (6.13), and using the relation (6.12), we obtain the final result

$$\left(\frac{\partial J}{\partial e}\right)_0=(E)_{t'}\geqslant 0 \tag{6.22}$$

This inequality should be satisfied for an arbitrary point t' not coinciding with corner points of the curve C. Nevertheless, continuity implies that it should be also satisfied at corner points.

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